

THE LOCAL-GLOBAL PRINCIPLE FOR INTEGRAL SODDY SPHERE PACKINGS

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ABSTRACT. Fix an integral Soddy sphere packing \mathcal{P} . Let \mathcal{K} be the set of all curvatures in \mathcal{P} . A number n is called *represented* if $n \in \mathcal{K}$, that is, if there is a sphere in \mathcal{P} with curvature equal to n . A number n is called *admissible* if it is everywhere locally represented, meaning that $n \in \mathcal{K} \pmod{q}$ for all q . It is shown that every sufficiently large admissible number is represented.

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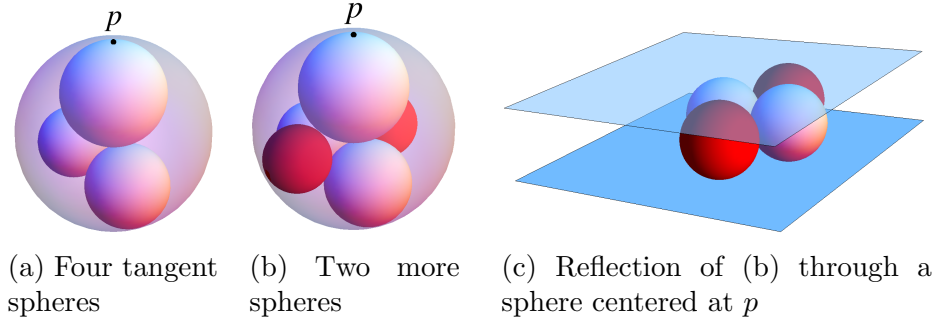


FIGURE 1

1. INTRODUCTION

This paper is concerned with a 3-dimensional analogue of an Apollonian circle packing in the plane, constructed as follows. Given four mutually tangent spheres with disjoint points of tangency (Figure 1a), a generalization to spheres of Apollonius's theorem says that

$$\text{there are exactly two spheres} \quad (1.1)$$

tangent to the given ones (Figure 1b). For a proof of (1.1), take a point p of tangency of two given spheres and reflect the configuration through a sphere centered at p . Thus p is sent to ∞ , and the resulting configuration (Figure 1c) consists of two tangent spheres wedged between two parallel planes; whence the two solutions claimed in (1.1) are obvious.

Returning to Figure 1b, one now has more configurations of tangent spheres, and can iteratively inscribe further spheres in the interstices (Figure 2a). Repeating this procedure *ad infinitum*, one obtains what we will call a *Soddy sphere packing* (Figure 2b).

The name refers to the radiochemist Frederick Soddy (1877-1956), who in 1936 wrote a *Nature* poem [Sod36] in which he rediscovered Descartes's Circle Theorem [Des01, pp. 37–50] and a generalization to spheres, see Theorem 2.3. The latter was known already in 1886 to Lachlan [Lac86], and appears in some form as early as 1798 in Japanese Sangaku problems [San]. We name the packings after Soddy because he was the first to observe that there are configurations of circle and sphere packings in which all curvatures¹ are integers [Sod37]; such a packing is called *integral*. The numbers illustrated in Figure 2b are

¹The curvature of a circle or sphere is one over its radius.

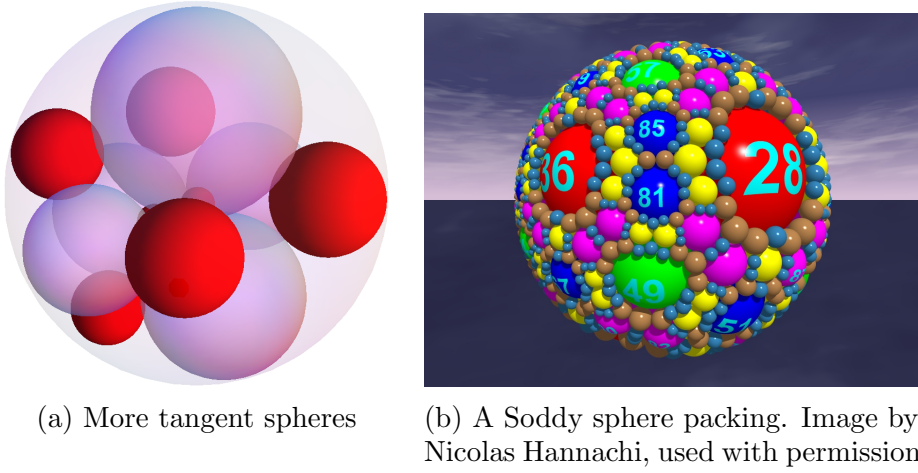


FIGURE 2

some of the curvatures in that packing. By rescaling an integral packing, we may assume that the only integers dividing all of the curvatures are ± 1 ; such a packing is called *primitive*. We focus our attention on bounded, integral, primitive Soddy sphere packings. In fact, all of the salient features persist if one restricts the discussion to the packing \mathcal{P}_0 illustrated in Figure 2b; the reader may wish to do so henceforth.

What numbers appear in Figure 2b? For a typical sphere $S \in \mathcal{P}$, let $\kappa(S)$ be its curvature, and let $\mathcal{K} = \mathcal{K}(\mathcal{P})$ be the set of all curvatures in \mathcal{P} ,

$$\mathcal{K} := \{n \in \mathbb{Z} : \exists S \in \mathcal{P}, \kappa(S) = n\}.$$

The bounding sphere is internally tangent to the others, so is given opposite orientation and negative curvature. The first few curvatures in \mathcal{P}_0 are:

$$\begin{aligned} \mathcal{K} = \{-11, 21, 25, 27, 28, 34, 36, 40, 42, 43, 46, 48, 49, 51, 54, 57, 61, \\ 63, 64, 67, 69, 70, 72, 73, 75, 78, 79, 81, 82, 84, 85, \dots\} \end{aligned} \quad (1.2)$$

A moment's inspection reveals that every curvature in \mathcal{P}_0 is

$$\equiv 0 \text{ or } 1 \pmod{3}, \quad (1.3)$$

that is, there are local obstructions. In analogy with Hilbert's 11th problem on representations of numbers by quadratic forms, we say that n is *represented* if $n \in \mathcal{K}$. Let $\mathcal{A} = \mathcal{A}(\mathcal{P})$ be the set of *admissible* numbers, that is, numbers n that are everywhere locally represented in the sense that

$$n \in \mathcal{K} \pmod{q} \text{ for all } q. \quad (1.4)$$

In our example, \mathcal{A} is the set of all numbers satisfying (1.3). The set of admissible numbers for any packing \mathcal{P} satisfies either (1.3) or

$$\equiv 0 \text{ or } 2 \pmod{3}, \quad (1.5)$$

see Lemma 2.11.

The number of spheres in \mathcal{P} with curvature at most N (counted with multiplicity) is asymptotically equal to a constant times N^δ , where δ is the Hausdorff dimension of the closure of the packing [Kim11]. Soddy packings are rigid (one can be mapped to any other by a conformal transformation), and so δ is a universal constant; it is approximately (see [Boy73a, BdPP94]) equal to

$$\delta \approx 2.4739 \dots$$

Hence one expects, on grounds of randomness, that the multiplicity of a given admissible curvature up to N is roughly $N^{\delta-1}$, which should be quite large. In particular, every sufficiently large admissible should be represented. The main purpose of this paper is to confirm this claim.

Theorem 1.6. *Integral Soddy sphere packings satisfy a local-to-global principle: there is an effectively computable $N_0 = N_0(\mathcal{P})$ so that if $n > N_0$ and $n \in \mathcal{A}$, then $n \in \mathcal{K}$.*

This theorem is the analogue to Soddy sphere packings of the local-global conjecture for integral Apollonian circle packings [GLM⁺03, FS11, BF11, BK12]. Being in higher dimension puts more variables into play, making the problem much easier.

For the proof, we study a certain infinite index subgroup Γ of the integral orthogonal group G preserving a particular quadratic form of signature $(4, 1)$. This group, Γ , which we call the *Soddy group*, is isomorphic to the group of symmetries of \mathcal{P} ; extended to act on hyperbolic 4-space, the quotient is an infinite volume hyperbolic 4-fold. Passing to the spin double cover of the orientation preserving subgroup of G , we find that Γ contains an arithmetic subgroup Ξ acting with finite co-volume on hyperbolic 3-space. A consequence is that the set \mathcal{K} of curvatures contains the “primitive” values of a shifted quaternary quadratic form (in fact an infinite family of such). Then using Kloosterman’s method, the local-global theorem follows. This generalizes to sphere packings the following related result in 2-dimensions due to Sarnak [Sar07]: the curvatures in an integral, primitive Apollonian circle packing contain the primitive values of a shifted binary quadratic form. It is in this sense that we have more variables: instead of binary forms, sphere packings contain values of quaternary forms.

Binary forms represent very few numbers, so despite some recent advances [BF11, BK12], the analogous problem in circle packings is wide open.

In dimension $n \geq 4$, one can start with a configuration of n tangent hyperspheres, repeating the above-described generating procedure. Unfortunately this does not give rise to a packing, as the hyperspheres eventually overlap [Boy73b]. Moreover there are no longer any such configurations in which all curvatures are integral (they can be S -integral, with the set S of localized primes depending on the dimension n); this follows from Gossett's [Gos37] generalization (also in verse) of Soddy's Theorem 2.3 to n -space.

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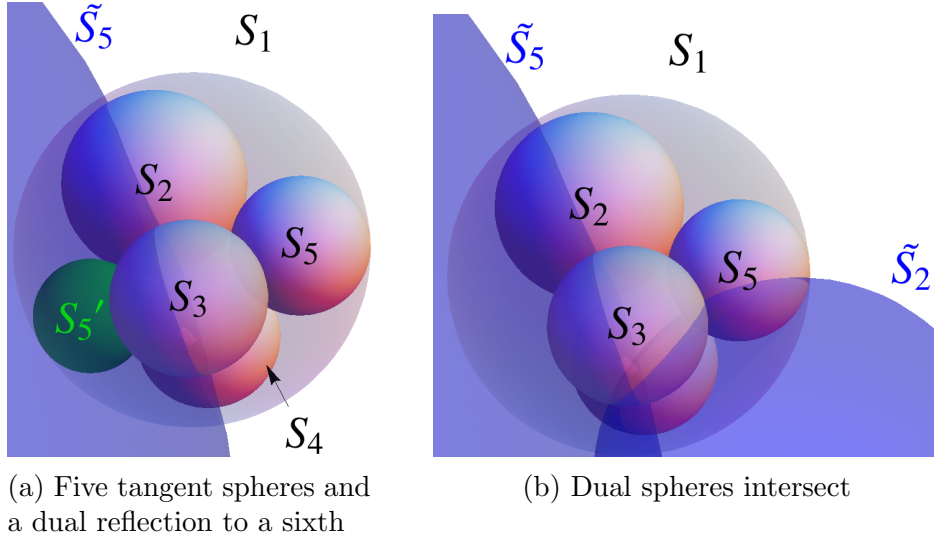


FIGURE 3

2. PRELIMINARIES

Let $\mathcal{S} = (S_1, S_2, S_3, S_4, S_5)$ be a configuration of five mutually tangent spheres, and let

$$\mathbf{v}_0 = \mathbf{v}(\mathcal{S}) = (\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$$

be the corresponding quintuple of curvatures, with $\kappa_j = \kappa(S_j)$. Any four tangent spheres, say S_1, S_2, S_3, S_4 have six cospherical points of tangency, and determine a *dual sphere* \tilde{S}_5 passing through these points. Similarly, for $j = 1, \dots, 4$, let \tilde{S}_j be the dual sphere orthogonal to all those in \mathcal{S} except S_j , and call $\tilde{\mathcal{S}} = (\tilde{S}_1, \dots, \tilde{S}_5)$ the *dual configuration*. Reflection through \tilde{S}_5 fixes S_1, S_2, S_3, S_4 , and sends S_5 to S_5' , the other sphere satisfying (1.1), see Figure 5a. The same holds for the other \tilde{S}_j , and iteratively reflecting the original configuration through the \tilde{S}_j *ad infinitum* yields the Soddy packing $\mathcal{P} = \mathcal{P}(\mathcal{S})$ corresponding to \mathcal{S} . Observe that unlike the Apollonian case, the dual spheres in $\tilde{\mathcal{S}}$ are not tangent, but intersect non-trivially, see Figure 3b.

Extend the reflections through dual spheres to hyperbolic 4-space,

$$\mathcal{H}^4 := \{(x_1, x_2, x_3, y) : x_1, x_2, x_3 \in \mathbb{R}, y > 0\}, \quad (2.1)$$

replacing the action of the dual sphere \tilde{S}_j by a reflection through a hyper(hemi)sphere \mathfrak{s}_j whose equator (at $y = 0$) is \tilde{S}_j (with $j = 1, \dots, 5$).

We abuse notation, writing \mathfrak{s}_j for both the hypersphere and the conformal map reflecting through \mathfrak{s}_j . The group

$$\mathcal{A} := \langle \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4, \mathfrak{s}_5 \rangle < \text{Isom}(\mathcal{H}^4), \quad (2.2)$$

generated by these reflections acts discretely on \mathcal{H}^4 . The \mathcal{A} -orbit of any fixed base point in \mathcal{H}^4 has a limit set in the boundary $\partial\mathcal{H}^4 \cong \mathbb{R}^3 \cup \{\infty\}$, which is the closure of the original sphere packing, a fundamental domain for this action being the exterior in \mathcal{H}^4 of the five dual hyperspheres \mathfrak{s}_j . Hence the quotient hyperbolic 4-fold $\mathcal{A} \backslash \mathcal{H}^4$ is geometrically finite (with orbifold singularities corresponding to non-trivial intersections of the dual spheres \tilde{S}_j), and has infinite hyperbolic volume with respect to the hyperbolic measure

$$y^{-4} dx_1 dx_2 dx_3 dy$$

in the coordinates (2.1). The group \mathcal{A} is the symmetry group of all conformal transformations fixing \mathcal{P} .

For an algebraic realization of \mathcal{A} , we need the following

Theorem 2.3 ([Lac86, Sod36]). *Given a configuration \mathcal{S} of five tangent spheres, the quintuple $\mathbf{v} = (\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$ of their curvatures lies on the cone*

$$Q(\mathbf{v}) = 0, \quad (2.4)$$

where Q is the quinternary quadratic form

$$Q(\kappa_1, \dots, \kappa_5) := 3(\kappa_1^2 + \dots + \kappa_5^2) - (\kappa_1 + \dots + \kappa_5)^2. \quad (2.5)$$

Recall again that a bounding sphere was negative curvature. Arguably the nicest formulation of Theorem 2.3 is the last line of the following excerpt from Soddy's aforementioned poem [Sod36].

To spy out spherical affairs / An oscular surveyor /
 Might find the task laborious, / The sphere is much the gayer, /
 And now besides the pair of pairs / A fifth sphere in the kissing shares. /
 Yet, signs and zero as before, / For each to kiss the other four /
 The square of the sum of all five bends / Is thrice the sum of their squares.

If $\kappa_1, \dots, \kappa_4$ are given, it then follows from (2.4) that the variable κ_5 satisfies a quadratic equation, and hence there are two solutions. This is an algebraic proof of (1.1). Writing κ_5 and κ'_5 for the two solutions, it is elementary from (2.4) that

$$\kappa_5 + \kappa'_5 = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4.$$

In other words, if the quintuple $(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$ is given, then one obtains the quintuple with κ_5 replaced by κ'_5 via a linear action:

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \kappa_5 \end{pmatrix} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \kappa'_5 \end{pmatrix}.$$

This is an algebraic realization of the geometric action of \tilde{S}_5 (or \mathfrak{s}_5) on a quintuple. Call the above 5×5 matrix M_5 . One can similarly replace other κ_j by κ'_j keeping the four complementary curvatures fixed, via the matrices

$$M_1 = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & & & & \\ 1 & -1 & 1 & 1 & 1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad (2.6)$$

$$M_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ 1 & 1 & -1 & 1 & 1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, M_4 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ 1 & 1 & 1 & -1 & 1 \\ & & & & 1 \end{pmatrix}.$$

Let Γ , isomorphic to \mathcal{A} , be the group generated by the M_j :

$$\Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle. \quad (2.7)$$

By construction, each generator M_j (and hence also Γ) lies inside the orthogonal group O_Q preserving the form Q ,

$$O_Q := \{g \in \mathrm{GL}_5 : Q(g \cdot \mathbf{v}) = Q(\mathbf{v}), \forall \mathbf{v}\}.$$

Moreover the Soddy group Γ is clearly contained in the group $O_Q(\mathbb{Z})$ of integer matrices. The fact that \mathcal{A} has infinite co-volume is equivalent to Γ having infinite index in $O_Q(\mathbb{Z})$. That is, Γ is a “thin” group. The generators of Γ satisfy the relations: $M_j^2 = I$ and $(M_j M_k)^3 = I$ [GLM⁺06, Theorem 5.1]. Geometrically, these relations correspond, respectively, to reflections being involutions, and to the non-trivial intersections of the dual spheres (recall Figure 3b).

The orbit

$$\mathcal{O} := \Gamma \cdot \mathbf{v} \quad (2.8)$$

of the quintuple $\mathbf{v} = \mathbf{v}(\mathcal{S})$ under the Soddy group Γ consists of all quintuples corresponding to curvatures of five mutually tangent spheres

in the packing \mathcal{P} . Hence the set \mathcal{K} of all curvatures in \mathcal{P} is simply the union of sets of the form

$$\mathcal{K} = \bigcup_{\mathbf{w} \in \{\mathbf{e}_1, \dots, \mathbf{e}_5\}} \langle \mathbf{w}, \Gamma \cdot \mathbf{v} \rangle, \quad (2.9)$$

as \mathbf{w} ranges through the standard basis vectors

$$\mathbf{e}_1 = (1, 0, 0, 0, 0), \dots, \mathbf{e}_5 = (0, 0, 0, 0, 1).$$

The inner product $\langle \cdot, \cdot \rangle$ in (2.9) is the standard one on \mathbb{R}^5 .

This explains the integrality of all curvatures in Figure 2b: the group Γ has only integer matrices, so if the initial quintuple \mathbf{v}_0 (or for that matter any curvatures of five mutually tangent spheres in \mathcal{P}) is integral, then the curvatures in \mathcal{P} are all integers (as first observed by Soddy [Sod37]).

From (2.9) it is elementary to see the local obstruction claimed in (1.3). For the packing \mathcal{P}_0 of Figure 2b, one can choose to generate from the “root” quintuple (meaning it consists of the curvatures of the five largest tangent spheres, see [GLM⁺03, §3])

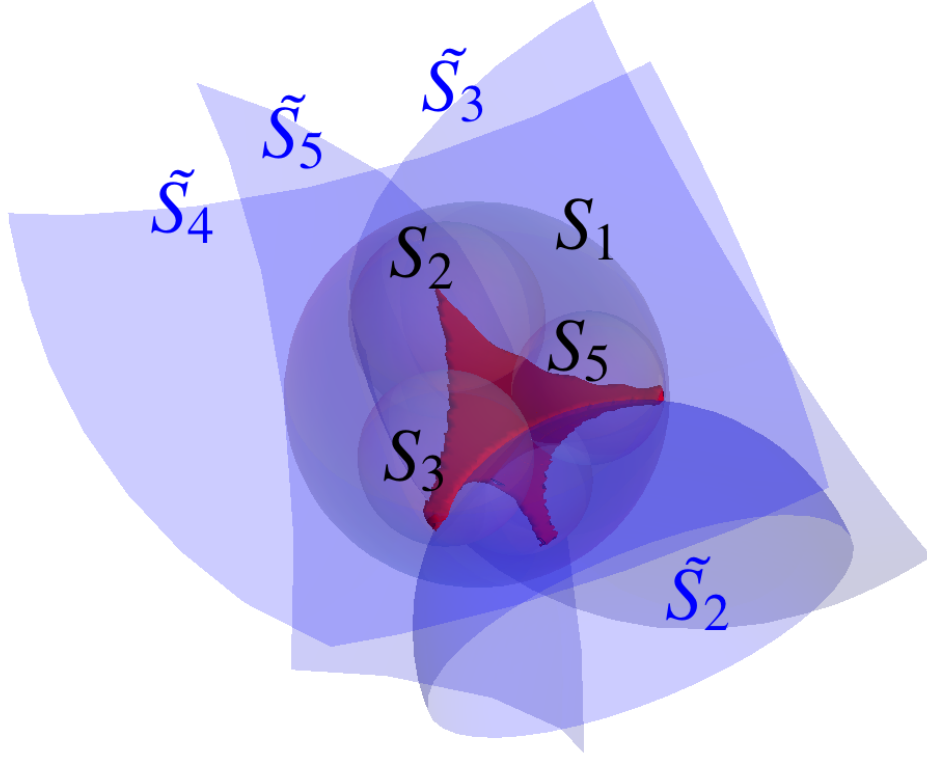
$$\mathbf{v}_0 := (-11, 21, 25, 27, 28). \quad (2.10)$$

The orbit under Γ , reduced mod 3 is then elementarily computed. In general we have the following

Lemma 2.11. *For \mathcal{K} the set of curvatures of an integral, primitive Soddy packing \mathcal{P} , there is always a local obstruction mod 3, either of the form (1.3) or (1.5). In particular, there is an $\varepsilon = \varepsilon(\mathcal{P}) \in \{1, 2\}$ so that, for any quintuple \mathbf{v} in the cone (2.4) over \mathbb{Z} , two entries are $\equiv 0 \pmod{3}$ and three entries are $\equiv \varepsilon \pmod{3}$.*

Note that we are not (yet) claiming that these are the *only* local obstructions; this will follow from a proof of the local-to-global theorem.

Proof. One may first attempt to understand the cone (2.4) over $\mathbb{Z}/3\mathbb{Z}$, but the form Q in (2.5) reduced mod 3 is highly degenerate. So instead consider the cone over $\mathbb{Z}/9\mathbb{Z}$. Disregarding the origin (since the packing is assumed to be primitive), there are 140 vectors mod 9, not counting permutations. Reducing these mod 3 leaves only the two vectors $(0, 0, \varepsilon, \varepsilon, \varepsilon)$, $\varepsilon \in \{1, 2\}$, and their permutations. The action of $\Gamma \pmod{3}$ on these is trivial: each vector is fixed. This is all verified by direct computation. \square

FIGURE 4. A fundamental domain for the action of \mathcal{A}_1

3. PROOF OF THEOREM 1.6

3.1. An Arithmetic Subgroup of the Soddy Group.

In this subsection, we prove that the Soddy group Γ , while being infinite index in $O_Q \cong O(4, 1)$, contains a subgroup which is arithmetic in $SO(3, 1)$, or alternatively, in its spin double cover $SL_2(\mathbb{C})$. The method is a generalization of Sarnak's in [Sar07].

Recall the configuration $\mathcal{S} = (S_1, \dots, S_5)$ of five mutually tangent spheres and the group \mathcal{A} in (2.2) of reflections through spheres in the configuration $\hat{\mathcal{S}}$ dual to \mathcal{S} . Let

$$\mathcal{A}_1 = \langle \mathfrak{s}_2, \dots, \mathfrak{s}_5 \rangle$$

be the subgroup of \mathcal{A} which fixes the sphere S_1 in \mathcal{S} . It acts discontinuously on the interior of S_1 , which we now consider as the ball model for hyperbolic 3-space \mathcal{H}^3 . A fundamental domain for the quotient $\mathcal{A}_1 \backslash \mathcal{H}^3$ is the curvilinear tetrahedron interior to S_1 and exterior to the dual spheres $\tilde{S}_2, \dots, \tilde{S}_5$, see Figure 4. In particular, it has finite volume.

To realize this geometric action algebraically, let

$$\Gamma_1 := \langle M_2, \dots, M_5 \rangle \quad (3.1)$$

be the corresponding subgroup of Γ , where the M_j are given in (2.6). We immediately pass to its orientation preserving subgroup, setting

$$\Xi := \Gamma_1 \cap \mathrm{SL}_5. \quad (3.2)$$

Then Ξ is generated by

$$\Xi = \langle \xi_1, \xi_2, \xi_3 \rangle, \quad (3.3)$$

where

$$\xi_1 := M_2 M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 2 & 2 \\ 1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \xi_2 := M_2 M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\xi_3 := M_2 M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

Lemma 3.4. *Let*

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & -1/2 & -1/2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then for $j = 1, 2, 3$, the conjugates

$$\tilde{\xi}_j := J \cdot \xi_j \cdot J^{-1} \quad (3.5)$$

are given by

$$\tilde{\xi}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & -1/2 & 0 \\ 0 & 0 & 3/2 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{\xi}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & -1/2 & 1/2 & -3/2 \\ 0 & 0 & -3/2 & -1/2 & 3/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (3.6)$$

and

$$\tilde{\xi}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -3/2 & -1/2 & 1/2 & 0 \\ 0 & 3/2 & -3/2 & -1/2 & 0 \\ 0 & 3 & 0 & -2 & 1 \end{pmatrix}.$$

Proof. Of course this can be verified by direct computation. But we elucidate the role of J as follows.

Let $\mathbf{v} = \mathbf{v}(\mathcal{S}) = (\kappa_1, \dots, \kappa_5)$ be the quintuple of curvatures corresponding to \mathcal{S} . Write the form Q in (2.5) as

$$\begin{aligned} Q(\kappa_1, \kappa_2, \dots, \kappa_5) &= 3(\kappa_1^2 + \kappa_2^2 + \dots + \kappa_5^2) - (\kappa_1 + \kappa_2 + \dots + \kappa_5)^2 \\ &= 2(\tilde{Q}(\mathbf{y}) + 3\kappa_1^2), \end{aligned}$$

where

$$\mathbf{y} = (y_2, \dots, y_5) := (\kappa_2, \dots, \kappa_5) + (\kappa_1, \kappa_1, \kappa_1, \kappa_1),$$

and

$$\tilde{Q}(\mathbf{y}) := y_2^2 + \dots + y_5^2 - y_2y_3 - y_2y_4 - \dots - y_4y_5.$$

The affine action of Ξ on $(\kappa_2, \dots, \kappa_5)$ is conjugated by the above to a linear action $\Xi' < \text{SO}_{\tilde{Q}}$. Since \mathbf{v} was assumed to be primitive, $\mathbf{y} = (\kappa_2 + \kappa_1, \kappa_3 + \kappa_1, \kappa_4 + \kappa_1, \kappa_5 + \kappa_1)$ is a primitive point on the quadric

$$\tilde{Q}(\mathbf{y}) = -3\kappa_1^2. \quad (3.7)$$

For later convenience we make another change of variables. Without loss of generality, assume that

$$\kappa_2 \equiv \kappa_3 \pmod{2}, \quad (3.8)$$

that is, κ_2 and κ_3 have the same parity; hence so do y_2 and y_3 . Let

$$y_2 = A - B - C + D, \quad y_3 = A + B - C + D, \quad y_4 = A, \quad y_5 = D,$$

or equivalently, set

$$A = y_4, \quad B = \frac{1}{2}(y_3 - y_2), \quad C = -\frac{1}{2}(y_2 + y_3) + y_4 + y_5, \quad D = y_5.$$

Then (3.7) becomes

$$F(\mathbf{a}) = -3\kappa_1^2,$$

where $\mathbf{a} = (A, B, C, D)$ and

$$F(\mathbf{a}) := 3B^2 + C^2 - 3AD. \quad (3.9)$$

The action Ξ' on \mathbf{y} is then conjugated to an action $\tilde{\Xi}$ on \mathbf{a} .

The matrix J is then simply the change of variables matrix from \mathbf{v} to \mathbf{a} . \square

The convenience of this conjugation is made apparent in the following

Lemma 3.10. *The quadratic form F in (3.9) has signature $(3, 1)$. Its special orthogonal group SO_F has spin double cover isomorphic to $\mathrm{SL}_2(\mathbb{C})$. There is a homomorphism $\rho : \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SO}_F$ given explicitly (for our purposes embedded in GL_5) by mapping*

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \quad (3.11)$$

to

$$\frac{1}{|\det(g)|^2} \begin{pmatrix} 1 & & & & \\ & |\alpha|^2 & \frac{2}{\sqrt{3}}\Im(\beta\bar{\alpha}) & \frac{2}{3}\Re(\beta\bar{\alpha}) & \frac{1}{3}|\beta|^2 \\ & \sqrt{3}\Im(\alpha\bar{\gamma}) & \Re(\delta\bar{\alpha} - \gamma\bar{\beta}) & \frac{1}{\sqrt{3}}\Im(\beta\bar{\gamma} + \alpha\bar{\delta}) & \frac{1}{\sqrt{3}}\Im(\beta\bar{\delta}) \\ & 3\Re(\gamma\bar{\alpha}) & \sqrt{3}\Im(\delta\bar{\alpha} + \beta\bar{\gamma}) & \Re(\delta\bar{\alpha} + \gamma\bar{\beta}) & \Re(\delta\bar{\beta}) \\ & 3|\gamma|^2 & 2\sqrt{3}\Im(\delta\bar{\gamma}) & 2\Re(\delta\bar{\gamma}) & |\delta|^2 \end{pmatrix}. \quad (3.12)$$

The preimages under ρ of the matrices $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3$ in (3.6) are $\pm \mathbf{t}_1, \pm \mathbf{t}_2, \pm \mathbf{t}_3$, respectively, where:

$$\mathbf{t}_1 = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}, \quad \mathbf{t}_2 = \begin{pmatrix} \omega^{-2} & 3\omega \\ 0 & \omega^2 \end{pmatrix}, \quad \mathbf{t}_3 = \begin{pmatrix} \omega & 0 \\ \omega^2 & \omega^{-1} \end{pmatrix}. \quad (3.13)$$

Here

$$\omega = e^{\pi i/3} = \frac{1 + \sqrt{-3}}{2}.$$

Proof. The signature of F is computed directly, and its spin group being $\mathrm{SL}_2(\mathbb{C})$ is a general fact in the theory of quadratic forms, see e.g. [Cas78]. We construct ρ explicitly as follows. Observe that the matrix

$$X := \begin{pmatrix} 3A & C + i\sqrt{3}B \\ C - i\sqrt{3}B & D \end{pmatrix}$$

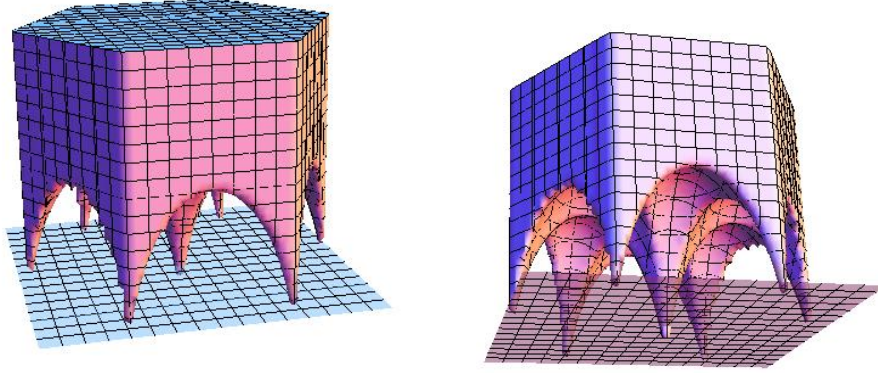
is Hermitian and has determinant $-F(\mathbf{a})$. Then for $g \in \mathrm{SL}_2(\mathbb{C})$,

$$X' := g \cdot X \cdot \bar{g}^t = \begin{pmatrix} 3A' & C' + i\sqrt{3}B' \\ C' - i\sqrt{3}B' & D' \end{pmatrix}$$

is also Hermitian with the same determinant. This gives a linear action sending (A, B, C, D) to (A', B', C', D') , which can be computed explicitly in the coordinates (3.11). The result (embedded in GL_5) is (3.12). The preimages (3.13) are then computed directly. \square

Let

$$\Lambda = \langle \pm \mathbf{t}_1, \pm \mathbf{t}_2, \pm \mathbf{t}_3 \rangle \quad (3.14)$$



(a) Top view

(b) Bottom view

FIGURE 5. A fundamental domain for the action of Λ in (3.14) on hyperbolic upper half space \mathcal{H}^3

be the group generated by (3.13). Another set of generators for Λ is

$$\mathbf{t}_1 = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix}, \quad \mathbf{t}_2^{-1} \cdot \mathbf{t}_1^{-4} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{t}_3^{-1} \cdot \mathbf{t}_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (3.15)$$

It is clearly a subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O}_3)$, where the Eisenstein integers $\mathcal{O}_3 = \mathbb{Z}[\omega]$ are the ring of integers of $\mathbb{Q}[\sqrt{-3}]$.

Lemma 3.16. *The group Λ is equal to $\Gamma_{0,\mathcal{O}_3}(3)$, where $\Gamma_{0,\mathcal{O}_3}(3)$ is the following Hecke congruence $\langle 3 \rangle$ -subgroup of $\mathrm{SL}_2(\mathcal{O}_3)$,*

$$\Gamma_{0,\mathcal{O}_3}(3) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_3) : \beta \equiv 0 \pmod{3} \right\}. \quad (3.17)$$

Proof. The inclusion

$$\Lambda \subset \Gamma_{0,\mathcal{O}_3}(3) \quad (3.18)$$

is clear from the generators (3.15). For the opposite inclusion, draw a Dirichlet domain for the fractional linear action of Λ on hyperbolic upper half space \mathcal{H}^3 using some short words in the generators (3.15). This fundamental domain is shown in Figure 5. The inclusion (3.18) induces the reverse inclusion

$$\Gamma_{0,\mathcal{O}_3}(3) \backslash \mathcal{H}^3 \subset \Lambda \backslash \mathcal{H}^3$$

on fundamental domains. A brute force search finds some small matrices satisfying (3.17):

$$\begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm 3\omega \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \pm 3\omega^{-1} \\ 0 & 1 \end{pmatrix}, \quad (3.19)$$

$$\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm\omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm\omega^{-1} & 1 \end{pmatrix}.$$

and then drawing another Dirichlet domain using these shows the equality of fundamental domains. Hence the groups are also equal. In hindsight, one can now directly verify that words in the generators (3.15) of Λ give the matrices in (3.19). \square

The point is that Λ is now seen to be an arithmetic group, so its elements can be parametrized, giving an injection of affine space into the otherwise intractable thin Soddy group Γ .

Proposition 3.20. *For any $\gamma, \delta \in \mathcal{O}_3$ with*

$$\gcd_{\mathcal{O}_3}(3\gamma, \delta) = 1, \quad (3.21)$$

there is an element

$$\xi_{\gamma, \delta} := J^{-1} \cdot \rho \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \cdot J \in \Xi < \Gamma_1 < \Gamma,$$

where $\xi_{\gamma, \delta} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 2\Re(\delta\bar{\gamma}) + 3|\gamma|^2 + |\delta|^2 - 1 & -\sqrt{3}\Im(\delta\bar{\gamma}) - \Re(\delta\bar{\gamma}) & \sqrt{3}\Im(\delta\bar{\gamma}) - \Re(\delta\bar{\gamma}) & 2\Re(\delta\bar{\gamma}) + 3|\gamma|^2 & 2\Re(\delta\bar{\gamma}) + |\delta|^2 \end{pmatrix}.$$

Proof. This follows from (3.17), (3.5), (3.3), (3.2) and (3.1). \square

3.2. Proof of the Main Theorem.

Recall that $\mathcal{O} = \Gamma \cdot \mathbf{v}$ in (2.8) is the orbit under the Soddy group Γ of a quintuple $\mathbf{v} = (\kappa_1, \dots, \kappa_5)$ of curvatures. If necessary, we may permute the κ_j to ensure that (3.8) holds. According to Lemma 2.11, there is an $\varepsilon = \varepsilon(\mathcal{P}) \in \{1, 2\}$ so that every curvature in \mathcal{K} is $\equiv 0$ or $\varepsilon \pmod{3}$.

Recalling that the set \mathcal{K} of curvatures contains sets of the form (2.9), and setting $\mathbf{w} = \mathbf{e}_5$, Proposition 3.20 immediately implies the following

Corollary 3.22. *The set \mathcal{K} of curvatures contains all the primitive values (meaning satisfying (3.21)) of the shifted quaternary quadratic form*

$$\begin{aligned}\mathfrak{F}_{\mathbf{v}}(\gamma, \delta) &:= \langle \mathbf{e}_5, \xi_{\gamma, \delta} \cdot \mathbf{v} \rangle \\ &= \kappa_1 (2\Re(\delta\bar{\gamma}) + 3|\gamma|^2 + |\delta|^2 - 1) + \kappa_2 \left(-\sqrt{3}\Im(\delta\bar{\gamma}) - \Re(\delta\bar{\gamma}) \right) \\ &\quad + \kappa_3 \left(\sqrt{3}\Im(\delta\bar{\gamma}) - \Re(\delta\bar{\gamma}) \right) + \kappa_4 (2\Re(\delta\bar{\gamma}) + 3|\gamma|^2) \\ &\quad + \kappa_5 (2\Re(\delta\bar{\gamma}) + |\delta|^2).\end{aligned}$$

Expanding $\gamma = \gamma_1 + \gamma_2\omega$, $\delta = \delta_1 + \delta_2\omega$ with $\gamma_j, \delta_j \in \mathbb{Z}$, we can write $\mathfrak{F}_{\mathbf{v}}$ as

$$\mathfrak{F}_{\mathbf{v}} = \mathbf{f}_{\mathbf{v}} - \kappa_1, \quad (3.23)$$

where $\mathbf{f}_{\mathbf{v}}$ is the homogeneous quaternary quadratic form

$$\begin{aligned}\mathbf{f}_{\mathbf{v}}(\gamma, \delta) &:= 3(\kappa_1 + \kappa_4)(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) + (\kappa_1 + \kappa_5)(\delta_1^2 + \delta_1\delta_2 + \delta_2^2) \\ &\quad + (\kappa_1 - 2\kappa_2 + \kappa_3 + \kappa_4 + \kappa_5)\gamma_1\delta_2 + (\kappa_1 + \kappa_2 - 2\kappa_3 + \kappa_4 + \kappa_5)\gamma_2\delta_1 \\ &\quad + (2\kappa_1 - \kappa_2 - \kappa_3 + 2\kappa_4 + 2\kappa_5)(\gamma_1\delta_1 + \gamma_2\delta_2).\end{aligned} \quad (3.24)$$

A calculation shows that $\mathbf{f}_{\mathbf{v}}$ is positive definite and has discriminant

$$\text{discr}(\mathbf{f}_{\mathbf{v}}) = \left(3 \left(\frac{1}{2}Q(\mathbf{v}) - 3\kappa_1^2 \right) \right)^2 = (3\kappa_1)^4, \quad (3.25)$$

by (2.4).

Proposition 3.26. *If n is sufficiently large with $(n, \kappa_1) = 1$ and $n \equiv 0$ or $\varepsilon \pmod{3}$, then n is represented in \mathcal{K} .*

Proof. Reducing (2.4), (2.5) mod 3 shows that $\kappa_1 + \dots + \kappa_5 \equiv 0 \pmod{3}$. Then reducing (3.24) mod 3 shows that $\mathbf{f}_{\mathbf{v}}(\gamma, \delta) \equiv (\kappa_1 + \kappa_5)\mathcal{N}(\delta)$, where $\mathcal{N}(\delta) = \delta_1^2 + \delta_1\delta_2 + \delta_2^2$. By (3.21), one can check that $\mathcal{N}(\delta) \equiv 1 \pmod{3}$, and hence $\mathbf{f}_{\mathbf{v}}(\gamma, \delta)$ is always

$$\equiv \kappa_1 + \kappa_5 \pmod{3}. \quad (3.27)$$

This is the only local obstruction for $\mathbf{f}_{\mathbf{v}}$.

Recall that $\kappa_1 \equiv 0$ or $\varepsilon \pmod{3}$, and assume the same holds for n . Assume $(n, \kappa_1) = 1$. This rules out the case $n \equiv \kappa_1 \equiv 0 \pmod{3}$. In the remaining cases, $n + \kappa_1 \not\equiv 0 \pmod{3}$, since the sum is either $\equiv \varepsilon$ or $2\varepsilon \pmod{3}$. By permuting $(\kappa_2, \dots, \kappa_5)$, we may arrange that

$$\kappa_5 \equiv n \pmod{3}. \quad (3.28)$$

(Lemma 2.11 guarantees such a κ_5 exists, regardless of κ_1 .) Then a permutation of $(\kappa_2, \kappa_3, \kappa_4)$ and pigeonhole ensures (3.8).

Then $n_1 := n + \kappa_1$ also has $(n_1, \kappa_1) = 1$, and $n_1 = n + \kappa_1 \not\equiv 0 \pmod{3}$. That is, n_1 is coprime to the discriminant (3.25). By (3.28), n_1 moreover satisfies the local condition (3.27). Kloosterman's method for quaternary quadratic forms (see e.g. [IK04, Theorem 20.9]) shows that every sufficiently large (with an effective bound) number which is locally represented by the form $\mathbf{f}_{\mathbf{v}}$ and coprime to its discriminant is also globally represented by $\mathbf{f}_{\mathbf{v}}$. Hence if n_1 is sufficiently large, it is represented by $\mathbf{f}_{\mathbf{v}}$. But then $n = n_1 - \kappa_1$ is represented by $\mathfrak{F}_{\mathbf{v}}$ in (3.23), and thus appears in \mathcal{K} by Corollary 3.22. \square

Since there is nothing special about κ_1 , Theorem 1.6 follows immediately from Proposition 3.26 by primitivity.

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